Wave attractors in anisotropic media

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1 density-stratified fluids, $g$
2 rotating fluids, $\Omega$
3 plasma’s, $B$
4 metamaterial, $\varepsilon$
Isotropic (2D) surface gravity waves:

Velocity potential

\[ e^{i(\kappa x + \lambda y - \omega t)} \rightarrow z = g \tanh(H), \]

\[ k = (\kappa, \lambda) = (\cos \alpha, \sin \alpha) \]

\[ \omega = \omega(\kappa) \rightarrow c_g = \nabla_k \omega \parallel c = \frac{\omega}{k^2}k \]

No constraint on direction!

\[ r = i, \rightarrow r = i \]

\[ u = (u, v) = \Box = i k = i(k, l) \]

\[ (u_i + u_r) \cdot n = 0 \]

\[ (k_i + k_r) \cdot n = 0 \rightarrow k_r = k_i \]

continuity of \( u_\parallel \)

Wave length conserved

\[ \alpha_r = \alpha_i \text{ specular reflection} \]

Wave ray divergence

Transient focusing

Berry 1987, ‘billiard dynamics’: Ray chaos
Anisotropic fluid

**Uniform stratification** \(N = \sqrt{-\frac{g}{\rho_0} \frac{d\bar{\rho}}{dz}} = \text{constant} \)

Heat & Salt => Density:

\[
\begin{align*}
\rho(z) &= \rho_0 + \rho(z) + \rho'(x,z,t), \\
0 \leq \max(-\rho(z)) \leq \max(\rho'(x,z,t))
\end{align*}
\]

**Visualisation mechanism:**
Light deflection due to changes in index of refraction, due to perturbations of density-stratification

Side view

Görtler 1943
Sakai, Iizawa, Aramaki 1997
Changing forcing frequency, $\omega$

\[ \omega = \omega(\alpha) = N \cos \alpha \]

$N = \text{constant}$

$\vec{c} = p \vec{k} - p \vec{g}$

$\vec{g} = (\cos \alpha, \sin \alpha)$

$\vec{k} = (\cos \alpha, \sin \alpha)$

$\omega = \omega(\alpha) = N \cos \alpha$

Frequency $\rightarrow$ angle, $\alpha$

Görtler 1943
Sakai, Iizawa, Aramaki 1997
Waves in continuously-stratified fluid

Oscillate at frequency $\omega$

Phase $c, k$

density contour displacements

Shearing currents, $u$

Wave energy, $c_g$
For fixed frequency, $\alpha$ is constant.

Horizontal upper-
And lower wall & vertical side wall

Sloping side wall
For fixed frequency \( \alpha \) is constant

*Geometric focusing*

*Maas & Lam 1995*
For fixed frequency $\alpha$ is constant

Geometric focusing

Courtesy: Jeroen Hazewinkel

Internal wave billiard
\[
\begin{align*}
\n\n\end{align*}
\]
Wave attractor properties

Multi-scale solutions of linear spatial wave equation, using nonlinear map of boundary onto itself, are selfsimilar in real space, parameter space and Fourier space.

Unobservable streamfunction:

$$\psi(x,z) = f(x+z) - g(x-z)$$

leads to amplified velocities (proportional to streamfunction derivatives) near focusing locations

$$u = (u, w) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)$$
Wave attractor properties

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\[ \psi(x, z) = f(x + z) - g(x - z) \]

leads to amplified velocities (proportional to streamfunction derivatives) near focusing locations.

\[ \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial z} \right) \psi = 0 \]
Wave attractor experiment

\[ G = g(1 + \varepsilon \sin 2\omega t) \]

Laser

Stratification \( N = \text{constant} = 1.89 \text{ s}^{-1} \)
\[ \omega = 1.44 \text{ s}^{-1} \]
Size: 261x261x96 mm
Amplitude: 10 cm

Maas, Benielli, Sommeria & Lam 1997
Dye displacement
subtracting initial lines

Oscillations start after \( \approx 5 \text{ min} \approx 50 \text{ oscillation periods} \)

Side view uniformly-stratified tank
Forcing by parametric excitation

Maas et al 1997
Shaking horizontally: growth phase

Viscous saturation: *Hazewinkel, v Breevoort, Dalziel & M. 2008*
Particle transport

Courtesy: Jeroen Hazewinkel
3 periods of oscillations, followed by stroboscopic view over many periods

Displacement of particles provides $u(x,y,t)$

Integrate kinematic equations
$$\frac{dx}{dt}=u(x,y,t)$$

trajectories virtual particles

See also: Beckebanze, Brouzet, Sibgatullin, Maas 2017
Wave attractor in Faroe Shetland channel?

Field observations: Isotherms (°C)
Vertical diffusivity: (green: low, red: high)
Model: (Curved) internal tidal rays

van Haren and van Raaphorst, 1999; Hosegood et al, 2005

Enhanced mixing and flows

Lab observation density perturbation

Courtesy: Jeroen Hazewinkel

Depth (m)

Gerkema, 2002
Initial Value Problem uniformly-stratified fluid

Initial stream function disturbance

Streamfunction: structure-preserving numerical method

NO attractor

Bajars, Frank, Maas 2013
Initial Value Problem uniformly-stratified fluid

Streamfunction: structure-preserving numerical method

- Initial stream function disturbance
- NO attractor
- Range of (1,1) attractors
- Frequency spectrum

Bajars, Frank, Maas 2013
Three-dimensional effects

Reflection of obliquely incident ray

Poincaré-Sobolev equation: \( P_{xx} + P_{yy} - P_{zz} = 0 \)

Thorpe 1997

Phillips 1963

Instantaneous refraction

\[
\sin \phi_r = \frac{s^2 - 1}{1 + 2s \cos \phi_i + s^2}
\]
Ray tracing in uniformly-stratified paraboloid

3D view

Focusing on wave attractor

Circle: critical depth

Top view

'Edge wave' type trapping

N=const

Maas 2005
Internal wave ray paths in uniformly-stratified parabolic channel
Internal tide generation in MICOM - dependence of cross-channel geometry

Cross-section:
Rectangular

Side view, mid-channel
Top view, half-depth

Drijfhout & Maas 2007
Internal tide generation in MICOM - dependence of cross-channel geometry

Cross-section:
- Rectangular
- Parabolic

Laboratory Exp Amplitude

Side view, mid-channel
Top view, half-depth
Transverse view

Drijfhout & Maas 2007

Lab confirmation: Pillet, Ermanyuk, Maas, Sibgatullin, Dauxois (2018)
Wave attractors in other anisotropic media?

Homogeneous, rotating fluid experiments

Lab experiment in trapezoidal channel, forcing by slight modulation of angular speed

Forcing by nutation of lid

In Geophysical and Astrophysical media
- Rotating fluids (inertial wave)
- Plasma’s subject to magnetic field (electron-cyclotron waves)

Mathematical expression:
\[ \omega = 2\Omega \sin \alpha \]

[Image of velocity magnitude]

Numerics: Planetary & Stellar interiors
Rieutord 2009

[Image of NEK5000 computation]

Maas 2001, Manders & Maas 2003

Sibgatullin, Ermanyuk, Maas, Xiulin, Dauxois 2017
Wave attractors in other anisotropic media?

Dipolar source

Natural, continuous Hyperbolic Metamaterial

$H = H_\perp$ acts as stream function $\psi$

Displacement (induction) $D = i \frac{\partial H}{\partial z} - \frac{\partial H}{\partial x}$

grows without bound...

Cellular pattern in ‘outer’ region

Hypboloidal Iso-Frequency Contour (IFC) in HMM

Large $k$, small wavelength

Spherical IFC in vacuum

$= (|k|)$

Ruben Maas 2016
Prescribe partial pressure in fundamental intervals

\[ \omega = N \cos \alpha \]

Summary: Basin shape matters!

Anisotropic media support waves that focus onto wave attractors: mixing locations, also attracting particles

Courtesy: Anna Rabitti
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